# Lecture 15: The Fibonacci Numbers

## COMS10007 - Algorithms

Dr. Christian Konrad

20.04.2020

# The Fibonacci Numbers

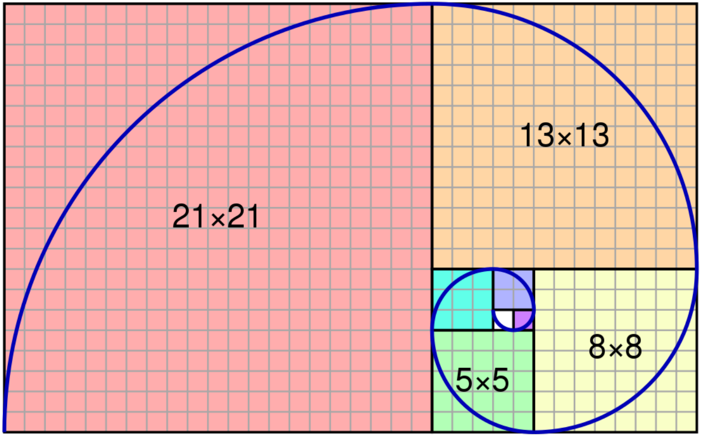
**Fibonacci Numbers**

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2 \,. \end{aligned}$$

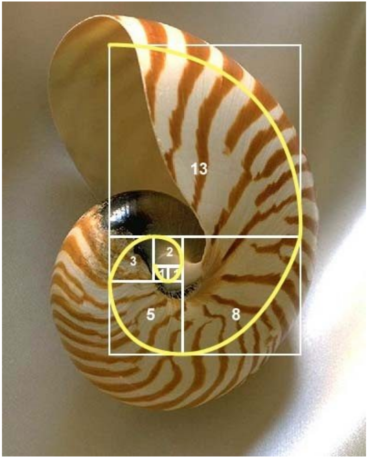
0 1 1 2 3 5 8 13 21 34 55 89 ...

**Why are they important?**

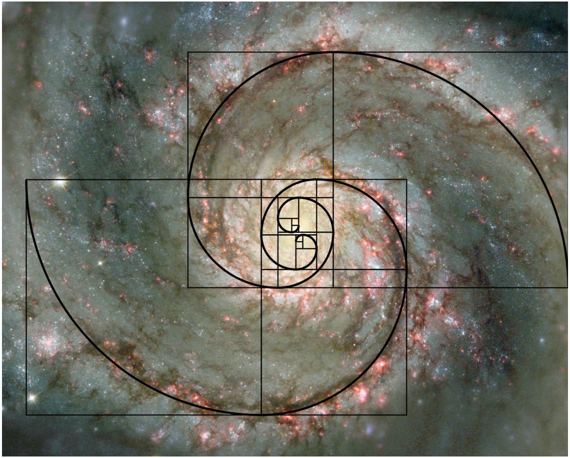
- Fibonacci heaps (data structure)
- Appear in analysis of algorithms (e.g. Euclid's algorithm)
- Appear everywhere in nature
- Interesting and instructive computational problem




source: wikipedia




source: realworldmathematics at wordpress



source: brian koberlein

# Computing the Fibonacci Numbers

**Naïve Algorithm**

> **Require:** Integer $n \geq 0$
>   **if** $n \leq 1$ **then**
>     **return** $n$
>   **else**
>     **return** $\textsc{Fib}(n-1) + \textsc{Fib}(n-2)$

$\textsc{Fib}(n)$

What is the runtime of this algorithm?

**Runtime:**

- Without recursive calls, runtime is $O(1)$
- Hence, runtime is $O($"number of recursive calls"$)$

# Runtime Analysis

**Define Recurrence:**

$T(n)$: number of recursive calls to $\textsc{Fib}$ when called with parameter $n$

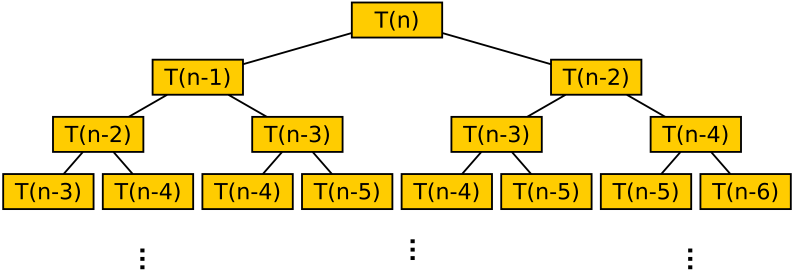
> **if** $n \leq 1$ **then**
>     **return** $n$
> **else**
>     **return** $\textsc{Fib}(n-1) + \textsc{Fib}(n-2)$

$$\begin{aligned} T(0) &= T(1) = 1 \\ T(n) &= 1 + T(n-1) + T(n-2) \text{ , for } n \geq 2 \quad . \end{aligned}$$

**How to Solve this Recurrence?**

- We will use the recursion tree technique to obtain a guess for an upper bound
- We will verify the guess with the substitution method

# Recursion Tree for $T$




**Observe:**

- Each node contributes 1
- Hence, $T(n)$ equals number of nodes
- Number of levels of recursion tree: $n$
- Our guess: $T(n) \leq c^n$ (we believe $c \leq 2$)

# Verification with the Substitution Method

**Recall:**

$$\begin{aligned} T(0) &= T(1) = 1 \\ T(n) &= 1 + T(n-1) + T(n-2)\ , \text{ for } n \geq 2\ . \end{aligned}$$

Our guess: $T(n) \leq c^n$

**Substitute Guess into Recurrence:**

$$T(n) = 1 + T(n-1) + T(n-2) \leq 1 + c^{n-1} + c^{n-2}$$

- It is required that $1 + c^{n-1} + c^{n-2} \leq c^n$
- The additive 1 prevents us from getting a similar form as $c^n$
- Try different guess: $T(n) \leq c^n - 1$

# Verification with the Substitution Method (2)

**New Guess:** $T(n) \leq c^n - 1$

$$\begin{aligned} T(n) &= 1 + T(n-1) + T(n-2) \\ &\leq 1 + (c^{n-1} - 1) + (c^{n-2} - 1) = c^{n-1} + c^{n-2} - 1 \ . \end{aligned}$$

**Select smallest possible** $c$:

$$\begin{aligned} c^{n-1} + c^{n-2} &= c^n \\ 0 &= c^2 - c - 1 \\ c &= \frac{1+\sqrt{5}}{2} \approx 1.618033989 \ . \text{ Golden Ratio!} \end{aligned}$$

**Base Case:**

- $T(0) = T(1) = 1$
- $c^0 - 1 = 0$ and $c^1 - 1 \approx 0.61$ ✗

# Verification with the Substitution Method (3)

**Another New Guess:** $T(n) \leq k \cdot c^n - 1$

$$\begin{aligned} T(n) &= 1 + T(n-1) + T(n-2) \\ &\leq 1 + (k \cdot c^{n-1} - 1) + (k \cdot c^{n-2} - 1) \\ &= k\left(c^{n-1} + c^{n-2}\right) - 1 \ . \end{aligned}$$

**Select smallest possible** $c$: $c = \frac{1+\sqrt{5}}{2}$ as before

**Base Case:**

- $T(0) = T(1) = 1$
- $k \cdot c^0 - 1 = k - 1$ and $k \cdot c^1 - 1 > k - 1$ ✓
- We can hence select $k = 2$!

We proved $T(n) \leq 2 \cdot (\frac{1+\sqrt{5}}{2})^n - 1$. Hence $T(n) \in O\left((\frac{1+\sqrt{5}}{2})^n\right)$.

# Fibonacci Numbers: Closed-form Expression

**Golden Ratio:**

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$$

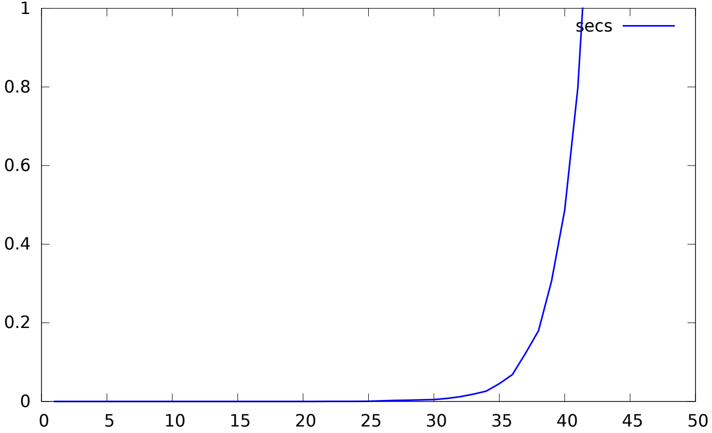
**Closed-form Expression:**

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} .$$

**Why not compute Fibonacci Numbers this way?**

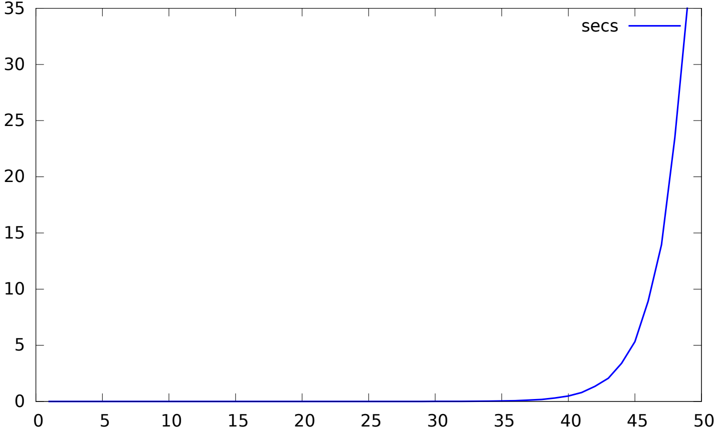
- Floating point operations, precision
- Large numbers involved
- Impractical

# Experiments



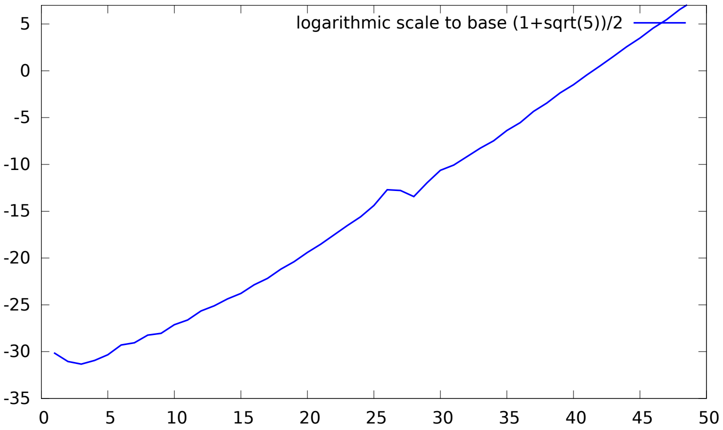

Exponential growth

# Experiments



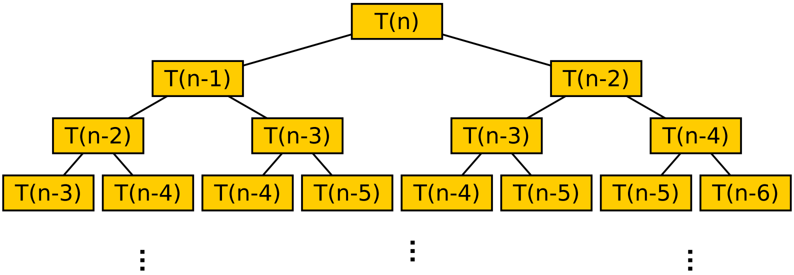

Exponential growth

# Experiments




Logarithmic scale

# Why is this Algorithm so slow?




**Discussion:**

- We compute solutions to subproblems many times ($T(i)$ is computed often, for most values of $i$)
- How can we avoid this?

**Dynamic Programming!**

# Dynamic Programming Solution

**Dynamic Programming** (will be discussed in more detail later)

- Store solutions to subproblems in a table
- Compute table bottom up

**Require:** Integer $n \geq 0$
  **if** $n \leq 1$ **then**
    **return** $n$
  **else**
    $A \leftarrow$ array of size $n$
    $A[0] \leftarrow 1$, $A[1] \leftarrow 1$
    **for** $i \leftarrow 2 \dots n$ **do**
      $A[i] \leftarrow A[i-2] + A[i-1]$
    **return** $A[n]$

$\textsc{DynPrgFib}(n)$

# Dicussion

**Analysis:**

- DynPrgFib() runs in time $O(n)$
- It uses space $\Theta(n)$ since it uses an array of size $n$

Can we reduce the space to $O(1)$?

**Improvement:**

- Observe that when $T(i)$ is computed, the values $T(1), T(2), \ldots, T(i-3)$ are no longer needed
- Only store the last two values of $T$

# Improved Algorithm

**Require:** Integer $n \geq 0$
  **if** $n \leq 1$ **then**
    **return** $n$
  **else**
    $a \leftarrow 0$
    $b \leftarrow 1$
    **for** $i \leftarrow 2 \dots n$ **do**
      $c \leftarrow a + b$
      $a \leftarrow b$
      $b \leftarrow c$
    **return** $c$

$\textsc{ImprovedDynPrgFib}(n)$

**Correctness:** via loop invariant!