Distributed Large Independent Sets in One Round On Bounded-independence Graphs

DISC 2015

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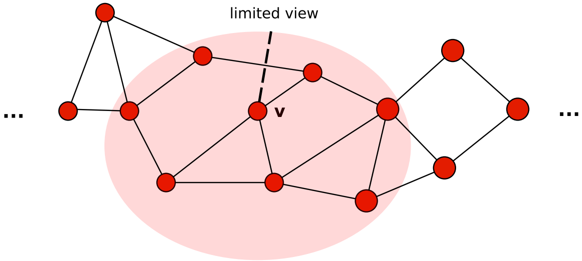
The $\mathcal{LOCAL}$ Model For Distributed Algorithms

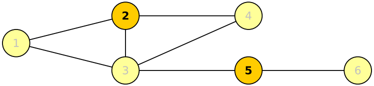
The $\mathcal{LOCAL}$ Model:

- Synchronous communication rounds along edges, individual messages of unbounded size
- Local computation is free
- Running time = maximal number of communication rounds
- Initially, nodes only know their degrees



Computing Independent Sets in the $\mathcal{LOCAL}$ Model

Output: After termination of algorithm, every node knows whether or not it joins the independent set



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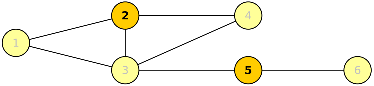

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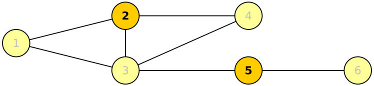

Are there good $\mathrm{O}(1)$ rounds Independent Set Algorithms?

- Maximality: $\Omega(\log^* n)$ rounds on ring **[Linial, 1992]**
- Maximum IS: $\mathrm{O}(1)$ rounds $\rightarrow \Omega(n^{\epsilon})$-approximation on general graphs **[Bodlaender, Konrad, Halldórsson, poster, DISC 2015]**

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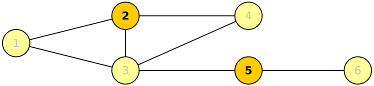

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Which graph classes admit poly-log approximations in $\mathrm{O}(1)$ rounds?

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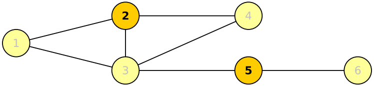

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Which graph classes admit poly-log approximations in $\mathrm{O}(1)$ rounds?
(e.g. $(1+\epsilon)$-approximation on planar graphs, **[Czygrinow et al., DISC 2008]**)

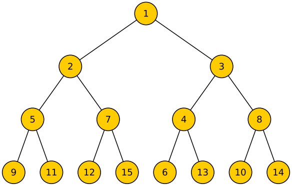
Bounded-independence Graphs

Definition: A graph $G = (V, E)$ is of *bounded-independence* if there exists a bounding function $f(r)$ so that for each node $v \in V$, the size of a maximum independent set in the r-neighborhood of v is at most $f(r)$.

r-**Neighborhood:** $\Gamma^r(v) = \{u \in V \setminus \{v\} : d(u,v) \leq r\}$.

Important: f is **independent** of n

Example: Binary Tree

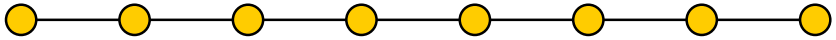


$f(r) \in \mathrm{O}(2^r)$

Bounded-independence Graphs (2)

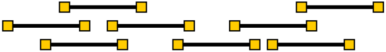
Further Examples:

- Path/Ring:



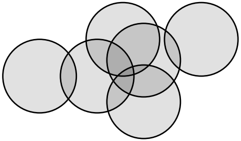
$$f(r) = r + 1$$

- Unit Interval Graphs: Intersection graph of unit intervals on line



$$f(r) = r + 1$$

- Unit Disc Graphs: Intersection graph of unit discs



$$f(r) \in \Theta(r^2)$$

Useful for Distributed Computing: **[Schn., Wattenh., Dist.Comp., 2010]**
Maximal IS and $(\Delta + 1)$-coloring in $O(\log^* n)$ rounds

Our Main Result

Let $G = (V, E)$ be of bounded-independence w.r.t. bounding function $f(r)$

Theorem: There is a randomized $\mathrm{O}(f(\frac{\log n}{\log \log n}))$-approximation algorithm for MIS on bounded-independence graphs with one communication round and single bit messages.

- Unit Interval Graphs: $\mathrm{O}(\frac{\log n}{\log \log n})$-approximation
- Unit Disc Graphs: $\mathrm{O}((\frac{\log n}{\log \log n})^2)$-approximation

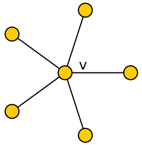
Put in Perspective: **[Schneider, Wattenhofer, Dist.Comp., 2010]**
$\mathrm{O}(1)$-approximation in $\mathrm{O}(\log^* n)$ rounds

Caro-Wei Bound

G is an n-vertex graph, independence number $\alpha(G)$

Caro and Wei:

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg_G(v) + 1} =: \beta(G).$$



Proof: **[Alon and Spencer]**

1. Every vertex v selects random number in $[0, 1]$
2. Put v into IS I if none of vs neighbors has chosen a larger number
3. $\Pr[v \in I] = \frac{1}{\deg_G(v)+1}$
4. $\mathbb{E}|I| = \sum_{v \in V} \Pr[v \in I] = \sum_{v \in V} \frac{1}{\deg_G(v)+1} = \beta(G)$. □

Distributed One-round Algorithm For Caro-Wei Bound:

- Nodes select values in $\{0, 1, \ldots, n^3\}$, broadcast to neighbors
- $\mathrm{O}(\log n)$ message sizes

Outline

1. **Approximation Guarantee**
   - Caro-Wei bound is good for graphs of *polynomially* bounded-independence

2. **Improved Algorithm**
   - Reducing message sizes to 1 bit
   - Algorithm works without knowledge of n

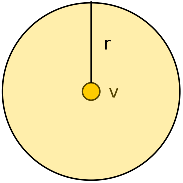
Caro-Wei bound is good for graphs of polynomially bounded-independence

Inverted Degree Sums

Small Neighborhood with Large Inverted Degree Sum

Let $v \in V$. What is the smallest $r \in \mathbb{N}$ so that:

$$\sum_{u \in \Gamma_G^r(v)} \frac{1}{\deg_G(u) + 1} = \Omega(1)\,?$$



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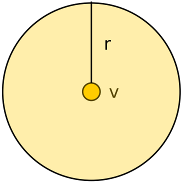

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Bounded-independence Graph:
Maximum independent set in r-neighborhood of $v \in V$ is at most $f(r)$

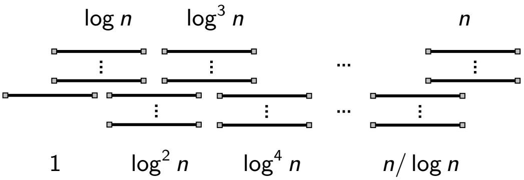
Hope: $r = \mathrm{O}(\mathrm{polylog}\, n)$ and f is a polynomial: $f(r) = \mathrm{O}(\mathrm{polylog}\, n)$

Inverted Degree Sums (2)

Theorem: $G = (V, E)$ arbitrary graph, and $v \in V$. Let $r = \frac{\log n}{\log \log n}$. Then:

$$\sum_{u \in \Gamma^r(v)} \frac{1}{\deg u} = \Omega(1).$$

Example: Interval graph (worst-case Example)



$$\begin{aligned}\sum_{v \in V} \frac{1}{\deg_G(v)} &\leq 1 \cdot \frac{1}{\log n} + \log n \cdot \frac{1}{\log^2 n} + \cdots + n/\log n \cdot \frac{1}{n} + n \cdot \frac{1}{n} \\ &= \frac{\log n}{\log \log n} \cdot \frac{1}{\log n} + 1 \leq 1 + o(1).\end{aligned}$$

Plugging it all together

Theorem: Caro-Wei gives a $O(f(\frac{\log n}{\log \log n}))$-approximation on polynomially bounded-independence graphs.

Proof idea:

- Pick greedily maximal set of center vertices $C \subseteq V$ so that every pair of vertices of C has mutual distance at least $2\frac{\log n}{\log \log n}$

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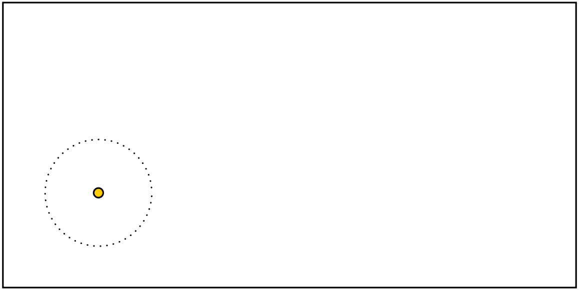

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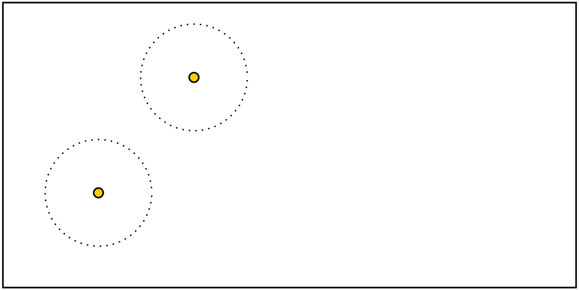

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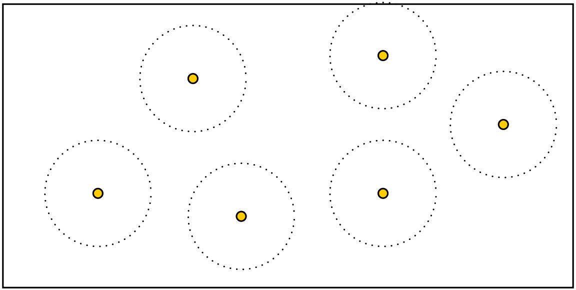

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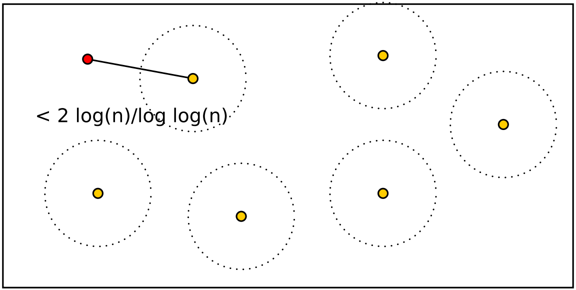

- Every $v \in V \setminus C$ is at distance at most $2\frac{\log n}{\log\log n}$ from a center vertex

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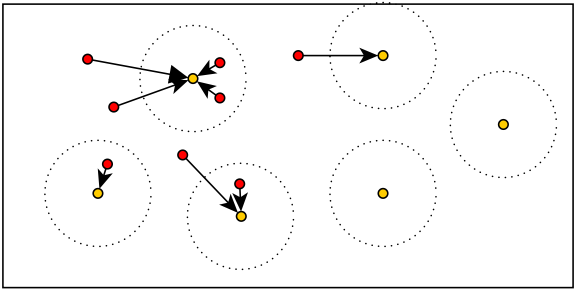


- Maximum independent set I^*: $|I^*| \le |C| f(2\frac{\log n}{\log\log n})$
- Algorithm: $\mathbb{E}\,|I| = \Omega(\sum_{v \in V} \frac{1}{\deg v}) = \Omega(|C|)$
- Approximation ratio: $\mathrm{O}(f(\frac{\log n}{\log\log n}))$ □

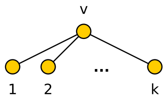
Improved Algorithm

Improved Algorithm

Algorithm: Every node $v \in V$:

1. Pre-selects itself with probability $\frac{1}{\deg v}$
2. Notifies neighbors whether v pre-selected itself (1 bit messages)
3. **if** pre-selected and no neighbor pre-selected **then**
   Join the independent set I

$$\begin{aligned}
\mathbb{E}\,|I| &= \sum_{v \in V} \Pr[v \in I] = \sum_{v \in V} \Pr[v \text{ pre-selected }] \cdot \Pr[v \in I \mid v \text{ pre-selected }] \\
&= \sum_{v \in V} \frac{1}{\deg_G(v)} \cdot \prod_{u \in \Gamma_G(v)} \underbrace{(1 - \frac{1}{\deg_G(u)})}_{=\Theta(e^{-\frac{1}{\deg_G(u)}})} \\
&= \sum_{v \in V} \frac{1}{\deg_G(v)} \cdot \Theta\left(e^{-\sum_{u \in \Gamma_G(v)} \frac{1}{\deg_G(u)}}\right) = \Theta\left(\sum_{v \in V} \frac{1}{\deg_G(v)}\right).
\end{aligned}$$



Caro-Wei: In $\mathrm{O}(1)$-claw-free graphs: $\sum_{u \in \Gamma_G(v)} \frac{1}{\deg_G(u)} = \mathrm{O}(1)$

Summary and Conclusion

Our Result

$\mathrm{O}(f(\frac{\log n}{\log \log n}))$-approximation in one round with one bit messages maximum independent set approximation in bounded-independence graphs

- Generalization? There are claw-free graphs so that Caro-Wei gives $\omega(\mathrm{polylog}\, n)$ approximation
- Multiple iterations (constant number)? Improvement by only a constant

Open Problems

- Is Caro-Wei the best we can do in one round?
- Randomly breaking ties with larger distance?

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