On the Power of Advice and Randomization for Online Bipartite Matching

ESA 2016

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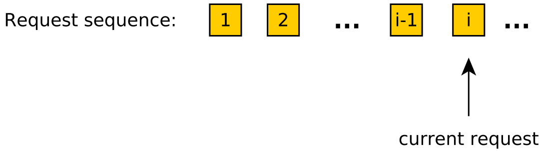
Outline

1. Online Algorithms: Randomization and Advice
2. Related Works and our Results
3. $(1 - \epsilon)$-approximation Algorithm
4. Ranking and Category Algorithms
5. Recent Developments

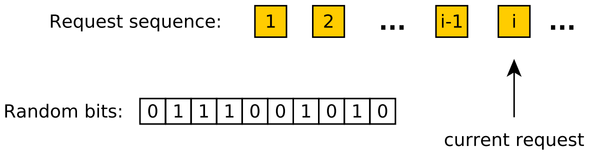
Online Algorithms: Randomization and Advice



Online Algorithms:

- Each request: Irrevocable decision

Online Algorithms: Randomization and Advice



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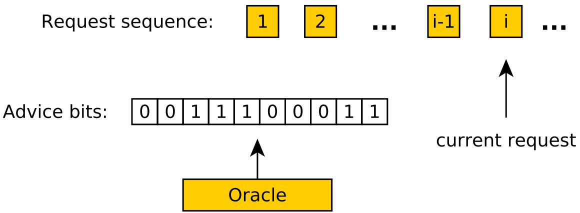
- Each request: Irrevocable decision
- Randomized online algorithm: Access to uniform random bits

Online Algorithms: Randomization and Advice



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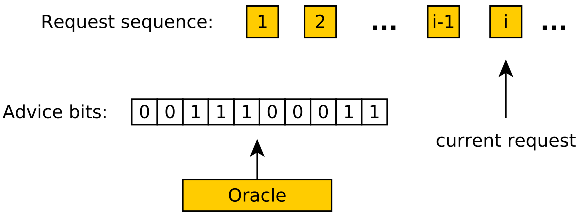

Online Algorithms:

- Each request: Irrevocable decision
- Randomized online algorithm: Access to uniform random bits
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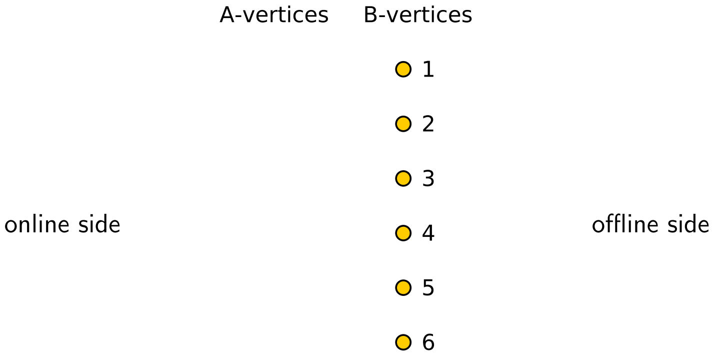
Research Question:

How helpful are advice bits as compared to random bits?

Online Bipartite Matching

$G = (A, B, E)$, $|A| = |B| = n$



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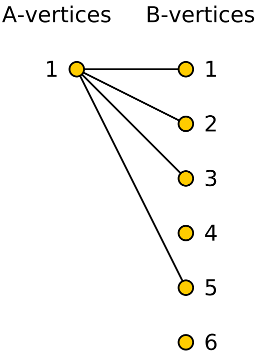
online side



offline side

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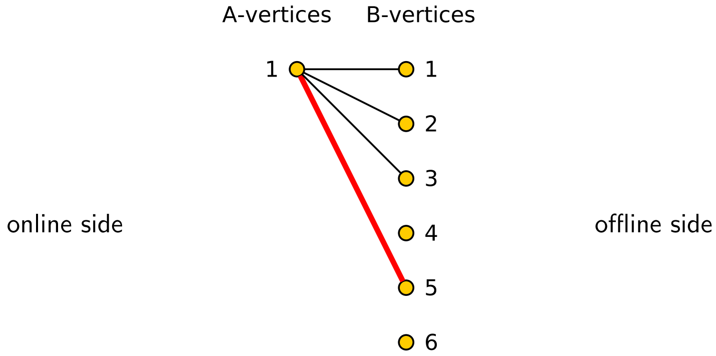

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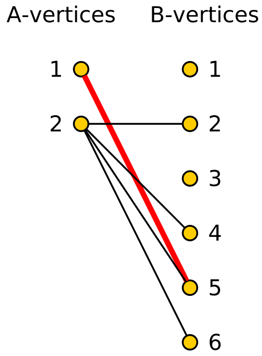

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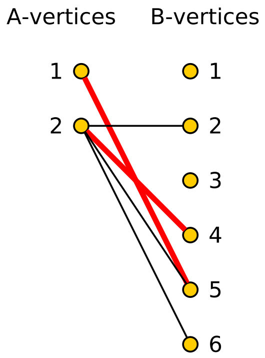

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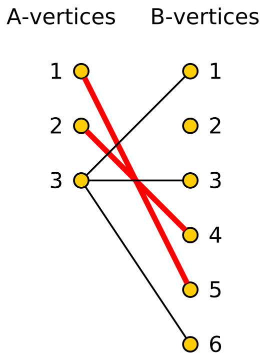

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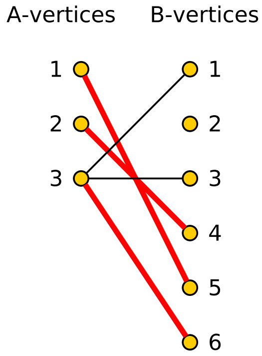

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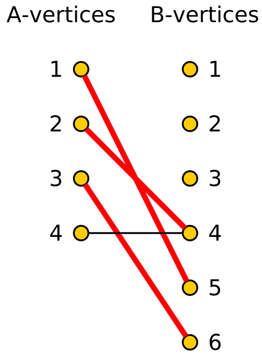

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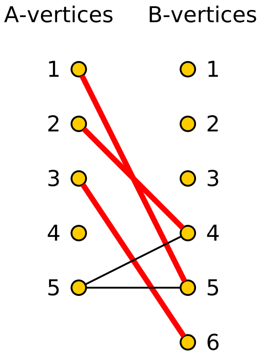

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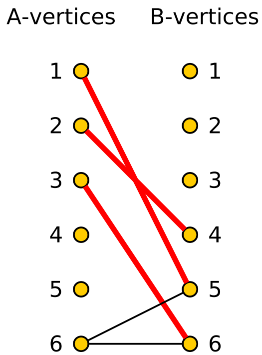

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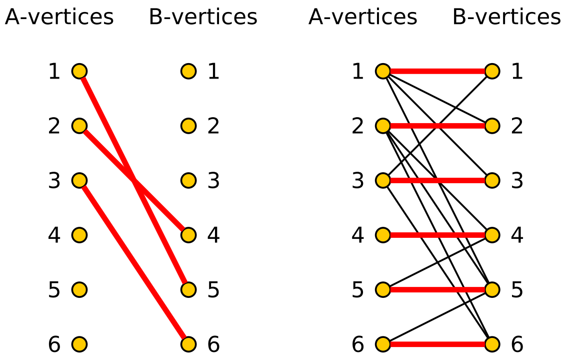


ALG: 3 **OPT:** 6

1/2-approximation

Online Bipartite Matching (2)

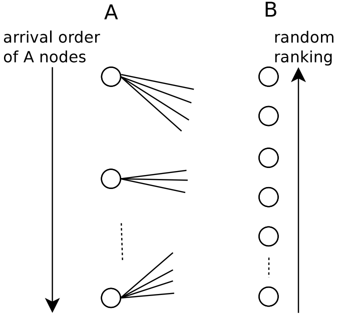
Algorithms:

- Matching a vertex if possible (Greedy): 1/2-approximation
- 1/2 is best possible for deterministic algorithms

- **[Karp, Vazirani, Vazirani, STOC 1990]**:
  Optimal randomized $(1 - 1/e)$-approximation (≈ 0.6321)

  1. $\pi :\leftarrow$ permutation of $1 \ldots |B|$, chosen uniformly at random

  2. Return $\textsc{Ranking}(\pi)$
  Match to lowest rank



Related Works and our Results

Known Results and Research Questions

$G = (A, B, E)$, $n = |A| = |B|$

Competitiveness	# of Adv./Rnd. Bits	Authors
Deterministic Advice Algorithms:		
1	$\Theta(n \log n)$	[Miyazaki, IPL, 2014]
$1 - 1/e + \epsilon$	$\Omega(n)$	[Mikkelsen, ICALP, 2016]
$1 - 1/e - \epsilon$	$\mathrm{O}(\log n)$	[Böckenhauer et al., ICALP, 2011]
$\frac{1}{2}$	0	GREEDY matching algorithm
Randomized Algorithms without Advice:		
$1 - 1/e$	$\mathrm{O}(n \log n)$	[Karp et al., STOC 1990]

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Indicating optimal match for each $a \in A$ is essentially optimal

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Constant size gadgets, each requiring at least one bit to improve on $1-1/e$

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Randomized Algorithms without Advice:		
$1 - 1/e$	$\mathrm{O}(n \log n)$	[Karp et al., STOC 1990]

- General method for transforming a randomized algorithm into a deterministic one with advice
- Exponential computation time of advice and algorithm

Known Results and Research Questions

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Chooses a permutation u.a.r., $\log(n!) = \mathrm{O}(n \log n)$ random bits

Known Results and Research Questions

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Questions:

1. Range $\omega(n) \cap o(n \log n)$: How much advice for $(1 - \epsilon)$-approx.?
2. Range $o(\log n)$: How much advice is needed to improve on $1/2$?
3. Randomized algorithm with fewer random bits?

Our Results

$G = (A, B, E)$, $n = |A| = |B|$

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$1-\left(\frac{2^k}{2^k+1}\right)^{2^k}$	kn	**[DKR]**

e.g. $5/9$-approximation with n random bits, or
$(1-1/e-0.0002)$-approximation with $10n$ random bits

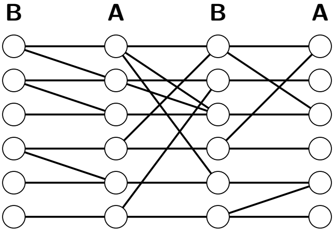
$(1-\epsilon)$-approximation Algorithm

Advice Algorithm

Streaming Algorithm [Eggert et al., ESA 2009]

- $(1-\epsilon)$-approximation using $\mathrm{O}(\frac{1}{\epsilon^5})$ passes over edge stream
- **In each pass** i**:** compute $\textsc{Greedy}$ matching M_i in vertex-induced subgraph G_i
- Final matching M is subset of $\bigcup_i M_i$

Example using three passes:



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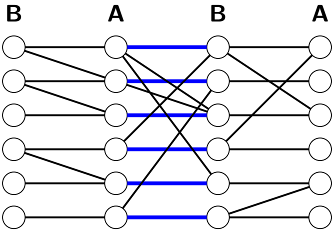

1. First pass M_1: $G_1 = G$

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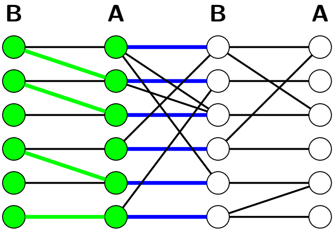

1. First pass M_1: $G_1 = G$
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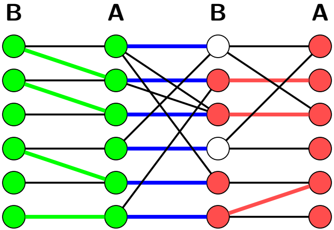

1. First pass M_1: $G_1 = G$
2. Second pass M_2: G_2 induced by ■ vertices
3. Third pass M_3: G_3 induced by ■ vertices

Advice Algorithm

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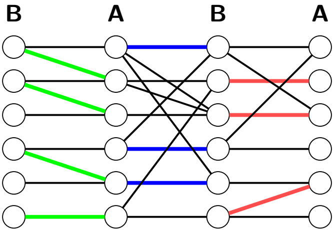
Example using three passes:



1. First pass M_1: $G_1 = G$
2. Second pass M_2: G_2 induced by ■ vertices
3. Third pass M_3: G_3 induced by ■ vertices
4. Compute M using edges from M_1, M_2 and M_3

Advice Algorithm (2)

Advice:

1. Graphs G_i using $\mathrm{O}(\frac{1}{\epsilon^5} n)$ bits
2. For each A-vertex: Index i s.t. M_i contains its match in M using $\mathrm{O}(\log(\frac{1}{\epsilon^5})n)$ bits

Algorithm:

1. Compute all matchings M_i in the background
2. Match A-vertex to partner as indicated by advice 2

$\rightarrow (1 - \epsilon)$-approximation using $\mathrm{O}(\frac{1}{\epsilon^5} n)$ advice bits

Lower Bound:

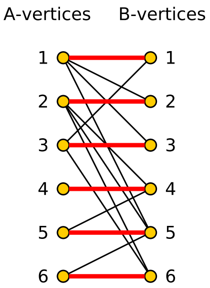
- String Guessing Game [Böckenhauer et al., Theo. Comp. Sci., 2014]
- Constant size semi-complete bipartite graphs as gadgets

Ranking and Category Algorithms

Ranking/Category Algorithms

Observation:

- Encoding the right permutation π_0:
  $\rightarrow$ $\textsc{Ranking}(\pi_0)$ gives optimal algorithm
- $n!$ permutations, $\rightarrow \log(n!) = \Theta(n \log n)$ bits



Definition (Ranking Algorithm):

1. Compute perm. π from advice/randomness
2. Return $\textsc{Ranking}(\pi)$

Definition (Category Algorithm):

1. Compute categories $\sigma : B \rightarrow \{0, \ldots, 2^b - 1\}$
2. Compute permutation π respecting σ and natural ordering
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Category algorithm uses nb bits of advice

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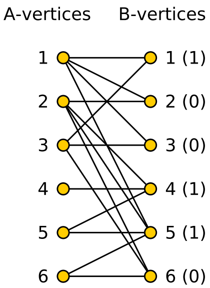

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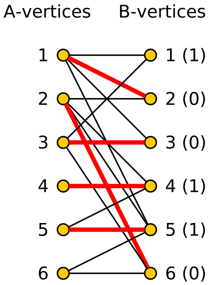

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Random Categories

Algorithm with reduced randomness:

1. Assign categories $\sigma : B \to \{0, \ldots, 2^b - 1\}$ randomly
2. Compute permutation π that respects σ and natural ordering
3. return $\textsc{Ranking}(\pi)$

Theorem: Approximation factor is $1 - \left(\frac{2^b}{2^b+1}\right)^{2^b}$

b	1	2	...	8	
Approx.	0.5555	0.5904	...	0.6314	$0.6321 (= 1 - \frac{1}{e})$

Analysis: Combination of the analyses of

- [Birnbaum, Mathieu, SIGACT News, 2008] for the KVV-algorithm, and
- [Konrad et al., APPROX, 2012] for a streaming algorithm

Remark: Assigning category 0 if $\textsc{Greedy}$ leaves vertex unmatched and 1 otherwise gives 0.6-approximation

Recent Developments

Recent Developments

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$1-1/e-\epsilon$	$O(\log n)$	**[DKR]**

Open Questions

Close Gaps:

- Advice needed for $(1-\epsilon)$-approximation:

UB	LB
$\mathrm{O}(\frac{1}{\epsilon^5}n)$	$\Omega(\log(\frac{1}{\epsilon})n)$

- Advice/Randomness needed for $(1/2+\epsilon)$-approximation:

UB	LB
$\mathrm{O}(\log n)$	$\Omega(\log \log n)$

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UB	LB
$O(\frac{1}{\epsilon^5}n)$	$\Omega(\log(\frac{1}{\epsilon})n)$

- Advice/Randomness needed for $(1/2+\epsilon)$-approximation:

UB	LB
$O(\log n)$	$\Omega(\log \log n)$

Thank you for your attention.